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On Periodic Solutions of the System $\dot{x} = y - F(x)$, $\dot{y} = -g(x)$*

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1. INTRODUCTION

The equation

$$\ddot{y} + \mu \sin \dot{y} + y = 0, \qquad \left(\cdot = \frac{d}{dt}\right) \tag{1}$$

or its equivalent representation

$$\dot{x} = y - \mu \sin x, \qquad \dot{y} = -x \tag{2}$$

furnishes an approximate description of the behavior of a second-order phase-locked loop, an electrical circuit used extensively in a variety of applications, notably in space communication.

The generalization of (2)

$$\begin{aligned} \dot{x} &= y - F(x), \\ \dot{y} &= -g(x), \end{aligned} \tag{3}$$

yields descriptions of the motion of Froude's pendulum, of the angular oscillations of a synchronous motor, and of the variation in electrical phase in a power distribution system, when F or g or both are taken to be oscillatory.

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Despite the interest in (2) and (3) occasioned by these applications, it is only recently that Hochstadt and Stephan [1] first proved the existence of an infinite number of limit cycles for (2) when $|\mu|$ is sufficiently small. D'heedene [2] then showed that (2) possesses an infinite number of periodic solutions for all real μ. Comstock [3] generalized D'heedene's results, replacing $\sin x$ by an odd oscillatory $F(x)$ which satisfies certain hypotheses, given in Section 2.

Here, we extend this work to system (3) without including, however, oscillatory $g(x)$. We prove, in Section 2, the existence of a denumerable set of periodic orbits when $F(x)$ satisfies Comstock's hypotheses and $g(x)$ is odd and suitably restricted. We turn, in Section 3, to even $F(x)$, odd $g(x)$, and show that under some relatively mild hypotheses, system (3) will possess a nondenumerable set of periodic solutions.

2. Periodic Solutions: $F(x)$ and $g(x)$ odd

Let
H_1 :

(a) $F(x) = -F(-x)$; $F(x) \in C^2(-\infty, \infty)$; with $F'(x) = f(x)$, $|F(x)| \leqslant A$, and $|f(x)| \leqslant B$ for all x, with $yF(y) > 0$ for y sufficiently small.

(b) $F(x)$ be oscillatory; with $\{z_k\}^\infty$ the sequence of positive zeros of $F(x)$ one requires $\int_{z_n}^{z_{n+1}} |F(x)|\, dx$ to be a nondecreasing function of n. (These are Comstock's [3] hypotheses concerning $F(x)$.)

Furthermore, assume
H_2 :

(a) $g(x) = -g(-x)$; $g(x) \in C^1(-\infty, \infty)$, and, for some a, $g'(x) \geqslant a > 0$ for all x.

(b) with $G(x) = \int_0^x g(u)\, du$, that there exists an $x^* \geqslant A/\sqrt{a}$ such that $G(x)/g(x) \geqslant 2A\{B + \sqrt{a}\}/a$ for all $x \geqslant x^*$.

Consider the simple closed convex curves defined by

$$\lambda(x, y, C) = \tfrac{1}{2}(y - C)^2 + G(x) = \text{const.}$$

The time rate of change of λ along a solution trajectory is given by $\dot{\lambda} = g(x)[C - F(x)]$. If $C = A$, then, since $|F(x)| \leqslant A$, solutions in $x \geqslant 0$ cross these ovals from their interiors to their exteriors. Similarly, the ovals $\lambda(x, y, -A) = \text{constant}$ are crossed toward their interiors, in $x \geqslant 0$.

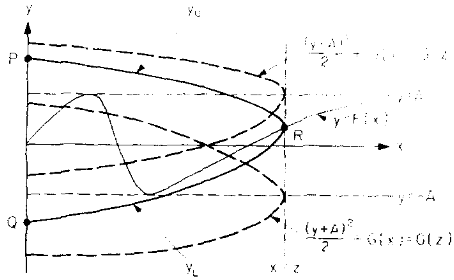

FIG. 1. The upper branch y_U and the lower branch y_L of a trajectory, together with two bounding ovals, are shown for $x \geqslant 0$.

We use these contours (which are analogous to the circles of D'heedene) to obtain the following bounds on the upper branch y_U and the lower branch y_L of a solution trajectory (see Fig. 1).

$$\begin{aligned} -A + \{2[G(z) - G(x)]\}^{1/2} &\leqslant y_U \leqslant A + \{2[G(z) - G(x)]\}^{1/2}, \\ -A - \{2[G(z) - G(x)]\}^{1/2} &\leqslant y_L \leqslant A - \{2[G(z) - G(x)]\}^{1/2}. \end{aligned} \tag{4}$$

Consider now the change in $\lambda(x, y, 0)$ as a solution proceeds from P to Q (Fig. 1). One has $(d/dt)\,\lambda(x, y, 0) = -g(x)\,F(x)$, so that $d\lambda = -(gF/y - F)\,dx$. Thus

$$\lambda(R) - \lambda(P) = \int_0^z - \frac{gF}{y_U - F}\,dx, \quad \text{and} \quad \lambda(Q) - \lambda(R) = \int_z^0 - \frac{gF}{y_L - F}\,dx.$$

Therefore,

$$\lambda(Q) - \lambda(P) = \int_0^z \left\{ \left(- \frac{gF}{y_U - F} \right) - \left(\frac{-gF}{y_L - F} \right) \right\} dx$$

or

$$\Delta(z) = \tfrac{1}{2}[y^2(Q) - y^2(P)] = \int_0^z F(x) \frac{d}{dx}(y_U - y_L)\,dx. \tag{5}$$

Notice that if for some value of z, $\Delta(z) = 0$, then the orbit passing through $(z, F(z))$ is periodic, for then $y(Q) = -y(P)$, and by the uniqueness, and mirror symmetry of trajectories through the origin, (when F and g are odd) the path will be closed.

Thus the proof of the existence of an infinite number of limit cycles now depends on demonstrating that $\Delta(z)$ has an infinite number of zeros.

Setting $d/dx(y_L - y_U) = h(x, z)$ we have $\Delta(z) = -\int_0^z F(x)\,h(x, z)\,dx$. Observe that $h(x, z) \geqslant 0$ for $0 \leqslant x \leqslant z$.

We need the result of D'heedene and Comstock (which we call Lemma I):

If $h(x, z)$, for fixed z, is a nondecreasing function of x (for $0 \leqslant x \leqslant z$) then

$$\operatorname{sign} \Delta(z_n) = (-1)^n.$$

The crucial part of the argument now lies in showing that $h(x, z)$ is monotone increasing in x, for then, by the continuity of $\Delta(z)$, a periodic orbit exists between successive zeros of $F(x)$.

THEOREM 1. *If H_1 is satisfied, and $g(-x) = g(x)$ with $g \in C^1(-\infty, \infty)$ and $4g'(x) \geqslant f^2(x)$ for all x, then* (3) *has an infinite number of limit cycles.*

Proof.

$$h(x, z) = \frac{d}{dx} y_L - \frac{d}{dx} y_U .$$

Now it is sufficient for

$$h' = y_L'' - y_U'' \geqslant 0$$

to have $y_L'' \geqslant 0$ and $y_U'' \leqslant 0$ separately.

Consider first

$$y_U'' = \frac{d}{dx}\left(-\frac{g}{y_U - F}\right) = -\left\{\frac{g'(x)\xi^2 + g(x)f(x)\xi + g^2(x)}{\xi^3}\right\},$$

where $\xi = y_U - F(x) \geqslant 0$ in $0 \leqslant x \leqslant z$.

Since $g^2(x) f^2(x) \leqslant 4g'(x) g^2(x)$ by hypothesis, it follows that the discriminant (of the quadratic in ξ) is negative, and therefore $y_U'' \leqslant 0$.

A similar argument shows that $y_L'' > 0$.

Remark. Notice that for (1) or (2), the condition $4g'(x) \geqslant f^2(x)$ becomes, $\mu^2 \leqslant 4$. In general, if a parameter μ is introduced, this condition may provide an upper bound on μ.

Unfortunately, the trajectories behave quite differently for large and for small μ; for example, $y_U'' \leqslant 0$ is not generally satisfied for μ large.

It was pointed out, however, by D'heedene and Comstock that $h'(x, z) \geqslant 0$ still holds for trajectories sufficiently remote from the origin.

THEOREM 2. *If H_1 and H_2 are satisfied, then* (3) *has an infinite number of limit cycles.*

Proof. It is sufficient to show that $h'(x, z) \geqslant 0$ for $z \geqslant x^*$, for then, by Lemma I, the theorem follows.

Now

$$h' = y_L'' - y_U'', \quad \text{and} \quad y_U' = -\frac{g}{y_U - F} = -g/\xi, \quad y_L' = -\frac{g}{y_L - F} = \frac{g}{\eta},$$

where $\xi = y_U - F \geqslant 0$ and $\eta = F - y_L \geqslant 0$.

Performing the differentiations, we obtain

$$h' = \frac{\xi + \eta}{\xi^3 \eta^3} [g^2(\eta - \xi)^2 + g^2\xi\eta + g'\xi^2\eta^2 + gf\xi\eta(\eta - \xi)].$$

Now h' will certainly be nonnegative if

$$\psi = g^2(x) + g'(x)\,\xi\eta + g(x)\,f(x)(\eta - \xi)$$

is nonnegative.

One has from (4), that

$$|\eta - \xi| = |y_U + y_L - 2F| \leqslant 4A.$$

Also

$$\int_0^x g(u)\,g'(u)\,du \geqslant a \int_0^x g(u)\,du = aG(x).$$

Therefore,

$$g^2(x) \geqslant 2aG(x). \tag{6}$$

On using $|f(x)| \leqslant B$, one gets

$$\psi \geqslant 2aG(x) + a\xi\eta - 4ABg(x). \tag{7}$$

We now require a lower bound on $\xi\eta$.

If x lies in R_1, the region where

$$[2\{G(z) - G(x)\}]^{1/2} \geqslant 2A$$

then (4) provides a positive lower bound on ξ and η, viz.,

$$\xi \geqslant [2\{G(z) - G(x)\}]^{1/2} - 2A,$$

and likewise for η. Therefore,

$$\xi\eta \geqslant 2[G(z) - G(x)] - 4A[2\{G(z) - G(x)\}]^{1/2} + 4A^2$$

or

$$\xi\eta \geqslant 2G(z) - 2G(x) - 4A[2G(z)]^{1/2} + 4A^2.$$

Using this in (7) yields

$$\psi \geqslant 2aG(z) - 4aA[2G(z)]^{1/2} - 4ABg(x) + 4aA^2.$$

Using $g(x) \leqslant g(z)$ and (6) in the form $g(z)/a^{1/2} \geqslant [2G(z)]^{1/2}$ results in

$$\psi \geqslant 2ag(z)\left[\frac{G(z)}{g(z)} - \frac{2A}{a}(B + \sqrt{a})\right] + 4aA^2$$

or

$$\psi > 0 \qquad \text{for} \quad z > x^*, \text{ by } H_2(\text{b}).$$

If, however, x lies in R_2, the region where

$$\{2[G(z) - G(x)]\}^{1/2} < 2A,$$

the lower bound obtained above, for ξ and η, is negative. In this region, we use

$$G(x) > G(z) - 2A^2, g(x) \leqslant g(z)$$

and $\xi\eta \geqslant 0$ to obtain, from (7),

$$\psi \geqslant 2a[G(z) - 2A^2] - 4ABg(z),$$

putting $\xi\eta = 0$ (since $\xi\eta \geqslant 0$).

On invoking H_2, viz.,

$$G(z) \geqslant \frac{2A}{a}(B + a^{1/2})\, g(z) \qquad \text{for} \quad z > x^*$$

yields

$$\psi \geqslant 4Aa^{1/2}g(z) - 4aA^2 = 4Aa^{1/2}[g(z) - a^{1/2}A].$$

Since $g(z) = \int_0^z g'(x)\, dx \geqslant az$, then

$$g(z) - a^{1/2}A \geqslant az - a^{1/2}A > 0,$$

whenever $z > A/a^{1/2}$, which is certainly satisfied for $z > x^*$.

Consequently, for orbits so large that $z > x^*$, then $h'(x, z) \geqslant 0$, whether x is in R_1 or in R_2, and sign $\Delta(z_n) = (-1)^n$, by Lemma I for $z > x^*$. Hence an infinite number of limit cycles exist

3. Periodic Orbits for $F(x)$ even, $g(x)$ odd

Let

h_1 :

(a) $F(x)$ satisfy a Lipschitz condition for every finite x interval,

(b) $F(x) = F(-x)$,

(c) $F(x) < 0 \quad \text{for} \quad x > 0$.

and h_2 :

(a) $g(x)$ satisfy a Lipschitz condition for every finite x interval,

(b) $xg(x) > 0 \quad \text{for} \quad x \neq 0$,

(c) $g(x) = -g(-x)$,

(d) $G(\pm\infty) = \infty$.

Theorem 3. *The system* (3) *possesses a nondenumerable set of periodic solutions, if the hypotheses h_1 and h_2 are satisfied.*

Proof. We first show that any solution trajectory of (3) starting on the negative y axis will intersect the y axis again. Next, we prove that the orbits described by (3) are symmetrical about the y axis, closed, hence periodic.

A solution starting at $(0, -y_0)$, $y_0 > 0$, will cross the closed contours $\lambda(x, y, 0) = C$ toward their interiors since $\dot{\lambda} = -g(x)F(x) \leqslant 0$ for $x \leqslant 0$.

Next, observe that the motion in reverse time $\tau = -t$, with $X = -x$, is governed by $X' = y - F(X)$, $y' = -g(X)(' \equiv d/d\tau)$, the same equations as (3). Thus the orbits described by (3) have mirror symmetry about the y axis. A solution beginning at $(0, -y_0)$ will intercept the y axis again at $(0, y_1)$ and return to $(0, -y_0)$, resulting in a closed, hence periodic orbit. It is clear that a periodic orbit passes through every point on the negative y axis. There are an uncountable number of limit cycles. Furthermore, it follows from the uniqueness of the solutions that the negative y axis is mapped by the orbits into a nondenumerable set of points on the positive y axis ($y_1 > 0$).

Corollary 3.1. *Let $h_1(c)$ be replaced by $h_1(c')$: $F(x) > 0$ for $x > 0$, all other hypotheses of Theorem* 3 *still holding, then Theorem* 3 *is still true. A periodic orbit will pass through each point on the positive y axis.*

Corollary 3.2. *Let $h_2(d)$ be replaced by $h_2(d')$: $G(\pm\infty) = Y_0^2$, $Y_0 > 0$, all other hypotheses of Theorem* 3 *still holding, then Theorem* 3 *is still true. A periodic solution will pass through each point in the interval* $(0, -Y_0)$ *on the negative y axis.*

AN EXAMPLE. The system

$$\dot{x} = y + x^{2n}, \qquad \dot{y} = -x^{2n-1} \quad (n \text{ a positive integer}) \tag{8}$$

satisfies all the conditions of Theorem 3. The corresponding phase equation can be integrated exactly; the solution trajectories are given by

$$x^{2n} = C\exp(-2ny) + \frac{1 - 2ny}{2n}. \tag{9}$$

If $-1/2n < C < 0$, the solutions are periodic, if $C \geqslant 0$ aperiodic. Periodic orbits intercept the negative y axis in $(-\infty, 0)$ and the positive y axis in $(0, 1/2n)$, whereas aperiodic orbits intercept the y axis once, in $(1/2n, +\infty)$.

THEOREM 4. *Let $h_1(c)$ be replaced by $h_1(c'')$: $|F(x)| \leqslant A$ for all x, all other hypotheses of Theorem 3 still holding, then the periodic orbits cover the $x - y$ plane.*

Proof. Starting with any initial point (x_0, y_0) in the plane the trajectory through that point is bounded by the oval

$$\lambda(x, y, -A) = \frac{(y + A)^2}{2} + G(x) = C \qquad \text{for} \quad x \geqslant 0$$

since $\dot{\lambda} = -g(x)[F(x) + A] \leqslant 0$ for $x \geqslant 0$, and by the closed nested ovals $\lambda(x, y, A) = C$ for $x \leqslant 0$, since $\dot{\lambda} = -g(x)[F - A] \leqslant 0$ for $x \leqslant 0$. The families of ovals $\lambda(x, y, -A) = C$, $\lambda(x, y, A) = C$ thus guide solutions to the axes.

It follows, from the mirror symmetry of orbits about the y axis, that a solution starting at $(x_0, 0)$, with $x_0 > 0$, will intercept the negative y axis at $(-x_0, 0)$, and return to $(x_0, 0)$, resulting in a closed, hence periodic, orbit. By proper choice of x_0, a periodic orbit can be made to pass through any point in the plane. Q.E.D.

COROLLARY 4.1. *Let $F(x)$ be oscillatory and let the hypotheses of Theorem 4 be satisfied. Then the periodic orbits cover the $x - y$ plane.*

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